# Algebra Qualifying Exam (May 2018)

1. (*10 points*) Let $F$ be a field. Show that the additive group $(F, +)$ can never be isomorphic to the multiplicative group $(F^{\times}, \cdot)$. (Hint: Consider orders of elements.)

2. (*10 points*) Let $p$ be a prime and suppose $K$ is a field of characteristic 0 such that for every proper finite extension $L/K$ the degree is divisible by $p$. Show that if $L/K$ is any finite extension of $K$, then $[L : K] = p^m$ for some $m \geq 0$. (Hint: Consider the Galois closure $E/K$ of $L$ and use the Sylow theorems.)

3. Let $p$ be a prime and $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$. For an integer $n$ relatively prime to $p$, let $\zeta_n \in \overline{\mathbb{F}}_p$ be a primitive $n$-th root of unity, i.e. $\zeta_n^n = 1$, but $\zeta_n^m \neq 1$ for any $0 < m < n$.

(a) (*3 points*) Show that the polynomial $x^n - 1 \in \mathbb{F}_p[x]$ is separable.

(b) (*5 points*) Explain why the extension $\mathbb{F}_p(\zeta_n)/\mathbb{F}_p$ is Galois and show that there is an injective group homomorphism $\mathrm{Gal}(\mathbb{F}_p(\zeta)/\mathbb{F}_p) \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^{\times}$.

(c) (*7 points*) Show that $[\mathbb{F}_p(\zeta_n) : \mathbb{F}]$ is equal to the order of $p$ in $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

4. Consider the polynomial $f(x) = x^3 - 5 \in \mathbb{Q}[x]$.

(a) (*8 points*) Let $F/\mathbb{Q}$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine $\mathrm{Gal}(F/\mathbb{Q})$ and describe all quadratic subextensions of $F/\mathbb{Q}$, i.e. all subfields $\mathbb{Q} \subset K \subset F$ with $[K : \mathbb{Q}] = 2$.

(b) (*7 points*) Show that the polynomial $f(x)$ is irreducible over $\mathbb{Q}(\sqrt{7})$ and determine $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{7}))$, where $L$ be the splitting field of $f(x)$ over $\mathbb{Q}(\sqrt{7})$.

(Recall that the discriminant of $x^3 + px + q$ is $D = -4p^3 - 27q^2$.)

5. (*10 points*) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$. Note that the tensor product is taken over $\mathbb{Z}$. (Hint: Consider the map $\mu\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, $(q_1, q_2) \mapsto q_1 q_2$.)

6. (*10 points*) Let $p$ be a prime. Show that $\mathrm{Ext}^n_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$ for all $n \geq 0$.